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ON INVARIANT TESTS FOR MULTIDIMENSIONAL NORMALITY

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Abstract. The paper deals with the analysis of some natural symmetries of one- and multidimensional problems of testing normality. Some groups of transformations are considered and their maximal invariants are found. An example of invariant testing multinormality is also given.

1. Introduction. Let $X = (X_1, \ldots, X_n)^T$ be a vector of observations. The independent random variables X_i are assumed to have the probability density function of the form $\sigma^{-1} f[(x-\mu)/\sigma]$. The null hypothesis H_0: $f = f_0$ against H_1: $f \in \mathscr{F}_1$ is to be checked. H_0 and H_1 are composite hypotheses. The parameters μ and σ are then the nuisance parameters and in most cases the uniformly most powerful test does not exist. So one restricts himself to some limited classes of tests and to some smaller family $\mathscr{F}_1$ to get the solution of such restricted problem which is, in some sense, optimal. An example of such a procedure is applying the invariance principle which makes use of natural symmetries of the problem. The basic notions and concepts connected with the invariance may be found in [3].

2. One-dimensional invariant tests. In one-dimensional problems the following groups of transformations are of particular interest:

$\mathscr{G}_1$ – the group of affine transformations $\boldsymbol{R}^n \to \boldsymbol{R}^n$ of the form $Y = aX + b(1, \ldots, 1)^T$, where $a \neq 0$ and $b \in \boldsymbol{R}$;

$\mathscr{G}_2$ – the group of transformations of the same form but with $a > 0$.

The problem of testing H_0 against H_1, as defined in Section 1, remains invariant with respect to $\mathscr{G}_2$. If we assume that f is an even function, the problem is invariant with respect to $\mathscr{G}_1$.

Let x_ν be the observation defined by

$$|x_\nu - \bar{x}| = \max_{1 \leqslant i \leqslant n} |x_i - \bar{x}|,$$

where $\bar{x}$ is the sample mean and let $\hat{\sigma}_x^2$ be the sample variance. Some maximal invariants under $\mathcal{G}_1$ and $\mathcal{G}_2$ are given in the two following theorems.

THEOREM 1. *The statistic*

$$T_1(X) = \operatorname{sgn}(x_\nu - \bar{x}) \left(\frac{x_1 - \bar{x}}{\hat{\sigma}_x}, \ldots, \frac{x_n - \bar{x}}{\hat{\sigma}_x} \right)$$

is a maximal invariant for the group $\mathcal{G}_1$.

Proof. Let $y_i = ax_i + b$ for $i = 1, \ldots, n$ and $a \neq 0$. Then

$$\begin{aligned} T_1(Y) &= \operatorname{sgn}[a(x_\nu - \bar{x})] \frac{a}{|a|} \left(\frac{x_1 - \bar{x}}{\hat{\sigma}_x}, \ldots, \frac{x_n - \bar{x}}{\hat{\sigma}_x} \right) \\ &= \operatorname{sgn}^2(a)\, T_1(X) = T_1(X) \end{aligned}$$

and T_1 is an invariant for $\mathcal{G}_1$. If $T_1(X) = T_1(Y)$, then

$$\operatorname{sgn}(y_\mu - \bar{y}) \frac{y_i - \bar{y}}{\hat{\sigma}_y} = \operatorname{sgn}(x_\nu - \bar{x}) \frac{x_i - \bar{x}}{\hat{\sigma}_x}, \qquad i = 1, \ldots, n,$$

and

$$y_i = \frac{\operatorname{sgn}(x_\nu - \bar{x})\hat{\sigma}_y}{\operatorname{sgn}(y_\mu - \bar{y})\hat{\sigma}_x} x_i + \bar{y} - \frac{\operatorname{sgn}(x_\nu - \bar{x})\hat{\sigma}_y}{\operatorname{sgn}(y_\mu - \bar{y})\hat{\sigma}_x} \bar{x} = ax_i + b,$$

$$i = 1, \ldots, n \text{ and } a \neq 0.$$

Hence T_1 is a maximal invariant for the group $\mathcal{G}_1$.

In a similar way one can prove the following

THEOREM 2. *The sample configuration*

$$T_2(X) = \left(\frac{x_1 - \bar{x}}{\hat{\sigma}_x}, \ldots, \frac{x_n - \bar{x}}{\hat{\sigma}_x} \right)$$

is a maximal invariant for the group $\mathcal{G}_2$.

For several f the explicit probability density functions of maximal invariants under $\mathcal{G}_1$ and $\mathcal{G}_2$ are known. For some f_0 and f_1 it leads to the most powerful invariant tests for testing $H_0\colon f = f_0$ against $H_1\colon f = f_1$. Important examples are the most powerful invariant tests for normality against uniform [7], Laplace [8], Cauchy [1] and exponential [7] alternatives. The well known Shapiro-Wilk test [6] and many other tests for normality are also invariant. Note that a good omnibus test for normality or a test against symmetric alternatives should be invariant with respect to $\mathcal{G}_1$ but seeking for a test sensitive for asymmetric alternatives of some particular type (e.g. distributions concentrated on intervals (a, ∞)), one should rather restrict himself to the group $\mathscr{G}_2$. An example of such a test is that for normality against the exponential alternative [7]. It is also sensitive for other asymmetric alternatives like log-normal and gamma distributions.

3. Multidimensional invariant tests. Let $X = [X_1, \ldots, X_n]$ denote now the n independent identically distributed random vectors $X_i \in \mathbf{R}^p$ having a probability distribution absolutely continuous with respect to the Lebesgue measure in $\mathbf{R}^p$. Let us consider the problem of testing H_0: $X_i \sim \mathscr{N}_p(m, \Sigma)$ against H_1: $X_i \sim \mathscr{P} \in \mathscr{F}_1$, where $\mathscr{F}_1$ is a family of nonnormal p-dimensional distributions.

One possible way of finding tests for such a problem is a generalization of one-dimensional tests to the multidimensional case. In paper [4] the Roy's intersection principle [5] has been applied to find multidimensional versions of some one-dimensional tests for normality. The Roy's principle leads to tests which are invariant with respect to the group G_1 of transformations of the form $Y = AX + B$, where A is a nonsingular $(p \times p)$-matrix and B is a $(p \times n)$-matrix with all its columns being identical. It reduces the composite hypothesis H_0 to the simple one and the critical regions of such tests do not depend on unknown parameters m and Σ. Just as in the one-dimensional case, the uniformly most powerful test does not exist and it is suitable to restrict the family of alternatives to get tests which are sensitive for departures from normality of some particular type. Such restricted problems may be not invariant with respect to G_1. If e.g. $\mathscr{F}_1$ is the class of distributions concentrated in the region included in $D = (a_1, \infty) \times \ldots \times (a_p, \infty)$, the testing problem is, in general, not invariant with respect to G_1. However, there exists a subgroup G_2 of G_1 for which testing H_0 against H_1 remains invariant.

Let

$$S_x = (X - \bar{X})(X - \bar{X})^T/n,$$

where

$$\bar{X} = \begin{bmatrix} \bar{x}_{1.} & \ldots & \bar{x}_{1.} \\ \ldots & \ldots & \ldots \\ \bar{x}_{p.} & \ldots & \bar{x}_{p.} \end{bmatrix}_{(p \times n)}, \qquad \bar{x}_{i.} = \frac{1}{n} \sum_{k=1}^{n} x_{ik},$$

be the sample covariance matrix and let a matrix L_x be defined by $S_x^{-1} = L_x L_x^T$. It may be e.g. the decomposition based on eigenvectors and eigenvalues or the triangular decomposition. Since the distributions of X_i are dominated by the Lebesgue measure, X is a full-rank matrix with probability 1 and, consequently, S_x is almost surely positive defined.

The following theorem is an extension of Theorem 1 to the multidimensional case:

THEOREM 3. *A maximal invariant for the group* G_1 *is the matrix* $B_x = M_x L_x^T (X-\bar{X})$, *where* M_x *is an orthogonal matrix defined in the proof.*

Proof. Let $B'_x = L_x^T(X-\bar{X})$ and $Y = CX+D$ be a transformation belonging to the group G_1. Then $Y-\bar{Y} = C(X-\bar{X})$, $S_y = CS_x C^T$, and, consequently,

$$S_y^{-1} = (C^T)^{-1} S_x^{-1} C^{-1} = (C^T)^{-1} L_x L_x^T C^{-1} = AA^T, \quad \text{where } A = (C^T)^{-1} L_x.$$

Since $S_y^{-1} = L_y L_y^T$, we have $L_y L_y^T = AA^T$. This implies the existence of an orthogonal matrix M such that $L_y^T = MA^T = ML_x^T C^{-1}$ and, further,

$$B'_y = L_y^T(Y-\bar{Y}) = ML_x^T C^{-1} C(X-\bar{X}) = MB'_x. \tag{1}$$

We call the matrix B'_x (B'_y) the *configuration of the sample* X (Y). Each column of B'_x (B'_y) is a point of $\boldsymbol{R}^p$ and will be called a *point of configuration* B'_x (B'_y). Since any transformation belonging to G_1 may change the configuration only according to (1), the Euclidean distances and angles between the points of configuration in $\boldsymbol{R}^p$ remain unchanged.

Now we indicate a new orthogonal coordinate system in $\boldsymbol{R}^p$ which is defined by the configuration itself. The orthogonal change of the coordinates will define the orthogonal matrix M_x desired in the theorem. Let us consider a new orthogonal coordinate system $\xi_1, \ldots, \xi_p$ with the same origin and the ξ_1-axis led through the point of configuration most distant from the origin. Then we project the remaining points of configuration onto the $(p-1)$-dimensional subspace $\mathscr{P}_{p-1} \subset \boldsymbol{R}^p$, orthogonal to ξ_1, and lead the ξ_2-axis through the projected point from $\mathscr{P}_{p-1}$ most distant from the origin. The remaining $n-2$ points from $\mathscr{P}_{p-1}$ are then projected onto the $(p-2)$-dimensional subspace $\mathscr{P}_{p-2} \subset \mathscr{P}_{p-1}$, orthogonal to ξ_2, and the ξ_3-axis is led through the projected point from $\mathscr{P}_{p-2}$ most distant from the origin. Analogously we define the axes $\xi_4, \ldots, \xi_p$ and, as a result, we get a new orthogonal coordinate system $\xi_1, \ldots, \xi_p$ which may be obtained from the previous one through an orthogonal transformation in $\boldsymbol{R}^p$. So we get the new coordinates of the points of configuration multiplying B'_x by an orthogonal matrix M_x which is defined by the change of coordinate system described above. Consequently, $B_x = M_x L_x^T(X-\bar{X})$ is an invariant for the group G_1.

If now $B_y = B_x$, we have $M_y L_y^T(Y-\bar{Y}) = M_x L_x^T(X-\bar{X})$. Hence

$$Y = (L_y^T)^{-1} M_y^T M_x L_x^T X + \bar{Y} - (L_y^T)^{-1} M_y^T M_x L_x^T \bar{X} = CX+D,$$

where

$$C = (L_y^T)^{-1} M_y^T M_x L_x^T \quad \text{and} \quad D = \bar{Y} - (L_y^T)^{-1} M_y^T M_x L_x^T \bar{X}.$$

It is easy to note that C is a nonsingular matrix and all columns of D are identical. Hence B_x is a maximal invariant for the group G_1.

The multidimensional version of the group $\mathscr{G}_2$ is the group G_2 of transformations of the form $Y = CX + D$, where C is an upper triangular matrix with positive diagonal and D is a matrix with all its columns being identical.

For a symmetric and positive defined matrix A we define its decomposition

$$A = LL^T, \tag{2}$$

where L is a lower triangular matrix with positive diagonal. Such a decomposition is unique [2].

Now we can formulate the multidimensional version of Theorem 2.

THEOREM 4. *A maximal invariant for the group G_2 is the configuration of the sample $B'_x = L_x^T(X - \bar{X})$, where L_x is defined by the decomposition* (2) *of the matrix S_x^{-1}.*

Proof. Let $Y = CX + D$ be a transformation belonging to G_2. Like above, we have

$$S_y^{-1} = (C^T)^{-1} L_x [(C^T)^{-1} L_x]^T = L_y L_y^T.$$

Note that $(C^T)^{-1} L_x$ is a lower triangular matrix. The uniqueness of the decomposition (2) implies that $L_y = (C^T)^{-1} L_x$. Hence we have

$$B'_y = L_y^T (Y - \bar{Y}) = L_x^T C^{-1} C (X - \bar{X}) = B'_x$$

and B'_x is invariant with respect to G_2.

Let now $B'_y = B'_x$ for two different matrices Y and X. Then we have $L_y^T(Y - \bar{Y}) = L_x^T(X - \bar{X})$. Hence $Y - \bar{Y} = (L_y^T)^{-1} L_x^T (X - \bar{X})$ and

$$Y = [(L_x^{-1})^T L_y^T]^{-1} X + \bar{Y} - [(L_x^T)^{-1} L_y^T]^{-1} \bar{X} = CX + D,$$

where

$$C = [(L_x^{-1})^T L_y^T]^{-1} \quad \text{and} \quad D = \bar{Y} - [(L_x^{-1})^T L_y^T]^{-1} \bar{X}.$$

Since C is an upper triangular matrix and all columns of D are identical, B'_x is a maximal invariant for G_2.

In order to describe some features of the group G_2 we need the following

Definition. The distribution of the p-dimensional random vector $X = (X_1, \ldots, X_p)^T$ will be called *left-bounded* if and only if:

1. the marginal distribution of X_p is bounded from the left, i.e. there exist a such that $a \leqslant X_p$ with probability 1;

2. the conditional distributions of $X_i | X_{i+1} = x_{i+1}, \ldots, X_p = x_p$ are bounded from the left for $i = 1, \ldots, p-1$ for any fixed values of $x_{i+1}, \ldots, x_p$.

Some characterization of the group G_2 is given by

THEOREM 5. *The group G_2 is the maximal subgroup of the group G_1 preserving left-boundedness of the distribution.*

Proof. Let $X=(X_1, \ldots, X_p)^T$ and $Y=(Y_1, \ldots, Y_p)^T$ be random vectors and $Y=AX+B$, where A is a $(p\times p)$-nonsingular matrix and $B=(b_1, \ldots, b_p)^T$. We assume the distribution of X to be left-bounded. It is sufficient to show that the distribution of Y is left-bounded if and only if A is an upper triangular matrix with positive diagonal.

Since $Y_p = a_{p,1}X_1+\ldots+a_{p,p}X_p+b_p$, Y_p is bounded from the left if and only if each term of the sum is bounded from the left. The marginal distributions of $X_1, \ldots, X_{p-1}$ may be unbounded, because the distribution of X may have nonzero density, e.g. in the region $D=\{X\colon X_p>a, X_{p-1}>-X_p, \ldots, X_1>-X_2-\ldots-X_p\}$. Hence each of the first $p-1$ terms is bounded from the left if and only if $a_{p,1}=\ldots=a_{p,p-1}=0$. Since X_p is bounded from the left, the p-th term is bounded from the left if and only if $a_{p,p}\geqslant 0$. Nonsingularity of the matrix A implies that $a_{p,p}>0$.

Further, we have $Y_{p-1}=a_{p-1,1}X_1+\ldots+a_{p-1,p-1}X_{p-1}+a_{p-1,p}X_p+b_{p-1}$ and we are seeking for conditions on which the conditional distribution $Y_{p-1}|Y_p$ is bounded from the left. Like above, all terms of the sum must be bounded from the left. Since $Y_p=a_{p,p}X_p$, the fixed value of Y_p fixes uniquely the value of X_p and the two last terms of the sum are fixed. By fixed value of X_p the variable X_{p-1} is, by definition, bounded from the left. Hence $a_{p-1,p-1}X_{p-1}$ is bounded from the left if and only if $a_{p-1,p-1}\geqslant 0$. Since the variables $X_1, \ldots, X_{p-2}$ may be, like above, unbounded, each of the first $p-2$ terms is bounded if and only if $a_{p-1,1}=\ldots=a_{p-1,p-2}=0$. This and the nonsingularity of A imply that $a_{p-1,p-1}>0$.

In the same way we show, step by step, that the rows $p-2, p-3, \ldots, 1$ have such a form that A is upper triangular matrix with positive diagonal. Analysing the form of the i-th row we use the fact, implied by previous steps of the proof, that fixed values of $Y_{i+1}, \ldots, Y_p$ fix uniquely the values of $X_{i+1}, \ldots, X_p$.

The distributions of the maximal invariants B_x and B'_x do not depend on the parameters m and Σ assuming that the population has the normal distribution $\mathcal{N}_p(m, \Sigma)$. This is due to the fact that for any fixed m and Σ (Σ is assumed to be positive defined) there exist transformations in G_1 and in G_2 transforming the $\mathcal{N}_p(0, I)$ distribution into the $\mathcal{N}_p(m, \Sigma)$ one.

Theorems 1 and 2 may be obtained from Theorems 3 and 4 assuming $p=1$. The sample variance matrix becomes then simply the sample variance, the matrix L_x there corresponds to $\hat{\sigma}^{-1}$, the one-dimensional matrix M_x reduces to $\operatorname{sgn}(x_\nu-\bar{x})$ and the multidimensional configuration of the sample becomes one-dimensional. The matrix M_x and the sgn-function play the same role: they introduce a new coordinate system defined by the sample itself.

The new coordinate system $\xi_1, \ldots, \xi_p$, described in the proof of Theorem 3, is not the only possible. One can obtain another coordinate system defined by the sample, just as for $p = 1$ one can take, e.g., $\operatorname{sgn}(x_1 - \bar{x})$ instead of $\operatorname{sgn}(x_\nu - \bar{x})$.

Similarly, one can define the right-boundedness of the multidimensional distribution and show that the group G_2 preserves it.

4. Invariant tests for multinormality. If we are seeking for tests for multinormality which are powerful against left-bounded alternatives, we should restrict ourselves to the invariance with respect to G_2. The class of tests being invariant with respect to G_2 is larger than those invariant with respect to G_1. Hence, usually there exist tests invariant with respect to G_2 and not invariant with respect to G_1 which for particular alternatives, not invariant for G_1 but invariant for G_2, have greater power than tests invariant with respect to G_1. It is plausible that such properties have for example the most powerful invariant tests for particular alternatives of the form

$$H_1\colon\ X_i \sim F[U(X-m)], \qquad m \in R^p, \tag{3}$$

U is a $(p \times p)$-upper triangular matrix and F is a distribution function of a left-bounded random vector $X \in R^p$.

The most powerful tests are, however, not available in an explicit form in the multidimensional cases because of technical difficulties in performing appropriate calculations (1). Therefore, in this section we propose some explicit tests invariant for G_2 and show by the Monte Carlo method that they perform in a reasonable way for two particular alternatives of the form (3) and have greater power than some tests known in the literature and being invariant with respect to G_1.

First we obtain a multidimensional version of the one-dimensional most powerful invariant test for normality against the exponential alternative derived in [7]. This test is also sensitive for other alternatives bounded from the left. We shall show, on examples, that our multidimensional version has analogous properties. We also propose new multidimensional versions of the Shapiro-Wilk test and investigate their power for some alternatives invariant with respect to G_2.

The one-dimensional statistic of the test [7] is $E = (\bar{x} - x_{(1)})/\hat{\sigma}_x$ and the Shapiro-Wilk test is based on statistic

$$W = \hat{\sigma}_x^{-1} \sum c_i x_{(i)}.$$

The coefficients c_i are tabulated in [6].

We propose the following method of construction of the multidimensional tests:

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(1) Such tests have been found only for two simple bivariate alternatives and have rather complicated form [9].

1° Compute the maximal invariant B_x for G_1 or B'_x for G_2.

It will be treated as a new transformed sample.

2° Compute the one-dimensional test statistic E or W for samples from "marginal distributions" (i-th row of the maximal invariant is an n-element sample from the i-th marginal distribution).

Note that computing these statistics is very simple because in each case the sample mean is zero and the sample variance is equal to one.

3° The multidimensional test statistic is then computed as the sum of squares of the marginal statistics.

Starting from the maximal invariant B_x (B'_x) we get the test which is invariant with respect to G_1 (G_2). Some additional arguments supporting the method presented above will be published in a separate paper.

Note that the multidimensional version of the Shapiro-Wilk test obtained in [4] is simply the one-dimensional test for the first margin of the transformed data forming the matrix B_x. To see this, note that the denominator of W^* in Section 3 of [4] may be expressed in our notation as

$$\max_{1 \leqslant j \leqslant n} \{(X_j - X^0)^T (nS_x)^{-1} (X_j - X^0)\}$$
$$= \frac{1}{n} \max_{1 \leqslant j \leqslant n} \{[L_x^T (X_j - X^0)]^T L_x^T (X_j - X^0)\} = \frac{1}{n} \|Y_\mu\|^2,$$

where X^0 is a column of the matrix $\bar{X}$ and Y_μ is the point of configuration most distant from the origin of the coordinate system. Analogously, the statistics u_j in [4] may be expressed as $u_j = Y_\mu^T Y_j / n$, where Y_j is the j-th point of configuration.

Let us write

$$v_j = \frac{u_j}{\frac{1}{\sqrt{n}} \|Y_\mu\|} = \frac{Y_\mu^T Y_j}{\sqrt{n} \|Y_\mu\|} = \frac{1}{\sqrt{n}} \|Y_j\| \cos \sphericalangle (Y_\mu, Y_j).$$

Then formulae (3.2) in [4] may be expressed as $W^* = \left[\sum a_j v_{(j)}\right]^2$, where $v_{(j)}$ are ordered values of v_j.

Since $\sum v_j = 0$ and $\sum v_j^2 = 1$, and $\sqrt{n} v_j$ are projections of the points of configuration onto the ξ_1-axis led through the point of configuration most distant from the origin, W^* is the one-dimensional Shapiro-Wilk test statistic for the first margin of the transformed data forming the matrix B_x.

Let us write:

EM – the multidimensional version of the test E invariant with respect to G_2;

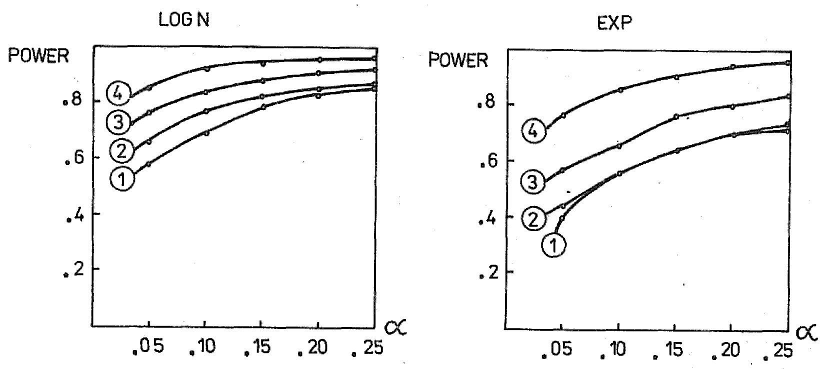
W^* – the multidimensional Shapiro-Wilk test described in [4];

WM_1 – the multidimensional Shapiro-Wilk test constructed the way described above and invariant with respect to G_1;

WM_2 – the version of WM_1 invariant with respect to G_2.

A preliminary Monte Carlo analysis of powers of these tests for the case $p = 2$ and $n = 10$ was accomplished. For each test 500 samples of size 10 from $\mathcal{N}_2(0, I)$ were generated and, for every sample, the test statistic was computed. Appropriate order statistics of the sampling distribution were used as empirical critical points. All critical regions were of the form $\{a\colon\ a < a_{cr}\}$. Two alternatives of the form (3) were considered with F being a distribution function of identically and independently distributed random variables. The marginal distribution was $\log N(0, 1)$ in the first and $\mathrm{EXP}(1)$ in the second case. For each alternative 500 samples of size 10 were generated and the test statistics were computed. As an estimate of the power of the tests, the fraction of samples for which the computed values of the test statistics have fallen into the critical region, was used. Figure A shows the results of

FIGURE A



① W^*, ② WM_1, ③ WM_2, ④ EM

computations. One can see that limiting the invariance to the group G_2 causes an essential rise of the power. The power of WM_2 is considerably greater than that of WM_1. Both tests WM_1 and WM_2 perform better than W^*.

The test EM has the best power against both alternatives. Like E in the one-dimensional case, EM seems to be a sensitive test against left-bounded nonsymmetric alternatives. For symmetric alternatives the power of EM may be, of course, smaller than that of W^*.

5. Some additional results. Let us now formulate some related results. Let G_p be the group of permutations of the sample, i.e., $gX = XP$ for $g \in G_p$, where P is a permutation matrix. The following theorems hold:

THEOREM 3′. *A maximal invariant for the group* (G_1, G_p) *is the matrix* $B_{(x)}$ *formed from* B_x *through ordering its columns with respect e.g. to the first component.*

THEOREM 4′. *A maximal invariant for the group* (G_2, G_p) *is the matrix* $B'_{(x)}$ *formed from* B'_x *through ordering its columns with respect, e.g., to the first component.*

The proofs of Theorems 3′ and 4′ follow from the proofs of Theorems 3 and 4, respectively, after replacing $Y = CX + D$ by $Y = CXP + D$.

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REFERENCES

[1] W. E. Franck, *The most powerful invariant test of normality versus Cauchy with applications to stable alternatives*, J. Am. Statist. Ass. 76 (1982), p. 1002–1005.

[2] P. Lancaster, *Theory of matrices*, New York 1969.

[3] E. L. Lehmann, *Testing statistical hypotheses*, New York 1959.

[4] J. F. Malkovich and A. A. Afifi, *On tests for multivariate normality*, J. Am. Statist. Ass. 68 (1973), p. 176–179.

[5] S. N. Roy, *Some aspects of multivariate analysis*, New York 1957.

[6] S. S. Shapiro and M. B. Wilk, *An analysis of variance test for normality*, Biometrika 52 (1965), p. 591–611.

[7] V. A. Uthoff, *An optimum property of two well-known statistics*, J. Am. Statist. Ass. 65 (1970), p. 1597–1600.

[8] – *The most powerful scale and location invariant test of the normal versus the double exponential*, Ann. Statist. 1 (1973), p. 170–174.

[9] Z. Szkutnik, *Most powerful invariant tests for binormality* (1986), to appear.

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